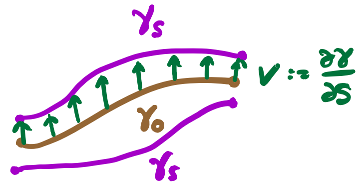
MATH 5061 Lecture on 4/1/2020

[Note: PS 4 due today, PS 5 posted and due on Apr 15.]

Recall: $(M, g) \rightsquigarrow f(p,q) := \inf_{\gamma} L(\gamma)$ metric space structure

$$\left[\frac{d}{ds}\Big|_{s=0} L(\gamma_s) = -\int_0^{\ell} \langle V, D_{\gamma'}\gamma' \rangle \, dt + \langle V, \gamma' \rangle \Big|_{t=0}^{t=\ell} \right]$$



1st variation formula

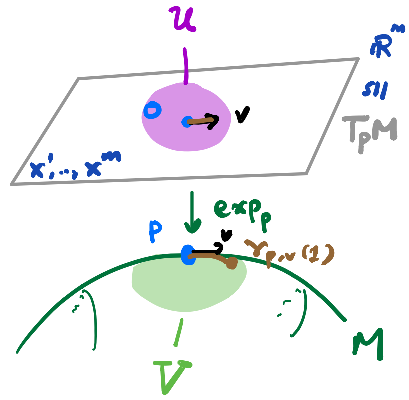
Geodesics $\Leftrightarrow$ $D_{\gamma'}\gamma' \equiv 0$ $\Leftrightarrow$ critical pts to L (w.r.t. fixed end points)

Exponential map of (M^m, g) at $p \in M$:

$$\exp_p : \overset{0}{\mathcal{U}} \subseteq T_pM \xrightarrow[\cong]{\text{local diffeo.}} \overset{p}{V} \subseteq M$$

$(\because d\exp_p(0) = Id_{T_pM})$

$$\exp_p(v) := \gamma_{p,v}(1)$$



Eucl. coord $x^1, \ldots, x^m$ on $T_pM \cong \mathbb{R}^m$ $\xrightarrow{\exp_p}$ geodesic normal coord. near $p \in M$.

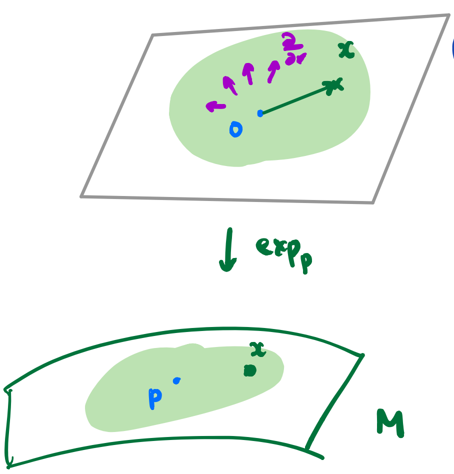
Properties of geodesic normal coord. $x^1, \ldots, x^m \rightsquigarrow g_{ij}, \Gamma_{ij}^k$

(1) $g_{ij}(0) = \delta_{ij}$ and $\Gamma_{ij}^k(0) = 0$.

[In geod. normal coord., $g_{ij}(x) = \delta_{ij} + \underbrace{O(|x|^2)}_{\text{curvature terms}}$]

(2) Gauss Lemma: $\boxed{\sum_{j=1}^m g_{ij}(x)\, x^j = x^i}$ — $(*)$

Geometric interpretation of $(*)$:



$(T_pM, g_{ij}) \rightsquigarrow \|x\|_g := [g(x,x)]^{1/2}$

$\cong$

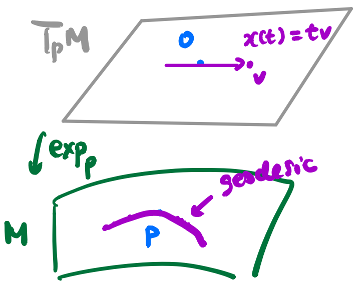
$(\mathbb{R}^m, \delta_{ij}) \rightsquigarrow r := |x| := [\delta(x,x)]^{1/2} = \left(\sum_{i=1}^m (x^i)^2\right)^{1/2}$.

$(*) \Leftrightarrow \boxed{\|dr\|_g \equiv 1}$

i.e. $\frac{\partial}{\partial r}$ is a unit vector field w.r.t g

$$\left[\because dr = \sum_i \frac{x^i}{|x|} dx^i \ ; \ \|dr\|_g^2 = \sum_{i,j} g^{ij}(x) \frac{x^i}{|x|} \frac{x^j}{|x|} = \frac{|x|^2}{|x|^2} = 1 \right]$$

"Proof": (1) $d\exp_p(0) = Id \Rightarrow g_{ij}(0) = \delta_{ij}$.



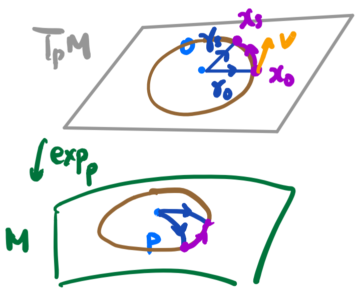
$t \mapsto \exp_p(tv)$ is a geodesic on M $\forall v \in T_pM$

geodesic eq$^{\underline{n}}$ $x(t) = tv$ $\Rightarrow$ $\cancel{\frac{d^2x^k}{dt^2}}^{\,0} + \Gamma_{ij}^k(x(t)) \underbrace{\frac{dx^i}{dt}}_{v^i} \underbrace{\frac{dx^j}{dt}}_{v^j} = 0$

At $t=0$ $\Rightarrow$ $\Gamma_{ij}^k(0) v^i v^j = 0$ $\forall v \in T_pM$

$\Rightarrow$ $\Gamma_{ij}^k(0) = 0$ for all i,j,k.

(2)



Fix $x_0 \in T_pM$, consider $\gamma_s : [0,1] \to M$ given in g.n.c. by

$$\gamma_s(t) = t\, x_s$$

where $s \mapsto x_s$ is a smooth curve on the "sphere" $\{ x \in T_pM : |x| = |x_0| \} \subseteq T_pM$.

Note: • For every fixed s,

(#) $\longrightarrow$ γ_s is a geodesic in M, with length $|x_0|$ and speed $= |x_0|$.

(##) $\begin{cases} \bullet\ \gamma_s(0) \equiv 0 \quad \forall s \\ \bullet\ \left.\frac{d}{ds}\right|_{s=0} \gamma_s(1) = \left.\frac{d}{ds}\right|_{s=0} x_s = v \end{cases}$

Notice: $v \perp x_0$ w.r.t. flat metric

ie $\sum_{i=1}^m v^i x_0^i = 0$

Idea: Apply 1st var. formula for L to $s \mapsto \gamma_s$.

$$0 \overset{(\#)}{=} \left.\frac{d}{ds}\right|_{s=0} L(\gamma_s) = -\int_0^1 \langle V, \underbrace{D_{\gamma_0'}\gamma_0'}_{\equiv 0 \text{ by } (\#)} \rangle_g \, dt + \underbrace{\langle V, \gamma_0' \rangle_g \Big|_{t=0}^{t=1}}_{\text{by } (\#\#),\ = \langle V, \gamma_0' \rangle_g (1)}$$

$\Rightarrow$ $\sum_{i,j} g_{ij}(x_0)\, x_0^i v^j = 0$ $\forall v$ s.t. $v \overset{\delta}{\perp} x_0$

$\Rightarrow$ $\sum_i g_{ij}(x_0)\, x_0^i \frac{\partial}{\partial x^j} \mathbin{/\!/} x_0^j \frac{\partial}{\partial x^j}$ w.r.t. flat metric δ

Also, $\sum_{i,j} g_{ij}(x_0)\, x_0^i x_0^j = \|\gamma_0'(1)\|_g^2 \overset{(\#)}{=} |x_0|^2 = \sum_i (x_0^i)^2 \Rightarrow \sum_i g_{ij}(x_0)\, x_0^i = x_0^j$

$\square$

Minimizing Properties of geodesics

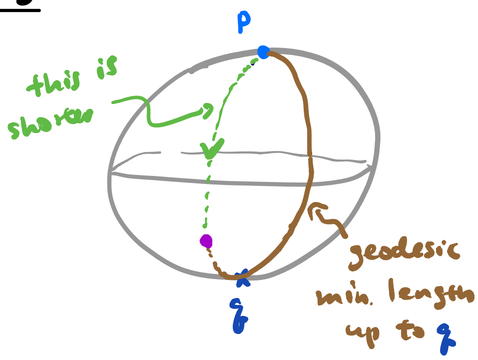
Recall: "geodesics" $\Longleftrightarrow$ "straight lines" in (M, g)

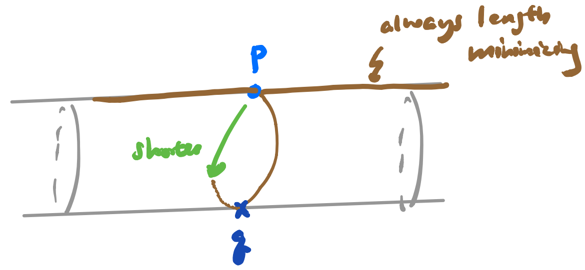
Q: How much does geodesics min. distance between 2 points?

FACT: "Short" geodesics minimizes length between its end points.

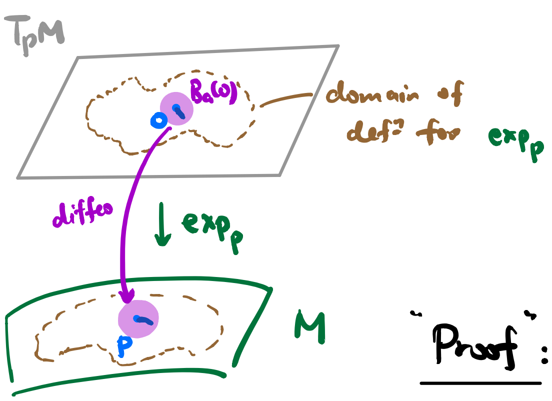
But not nec. true for "long" geodesics.

E.g.) $M = S^2 \subseteq \mathbb{R}^3$. $\qquad\qquad$ $M = S^1 \times \mathbb{R} \subseteq \mathbb{R}^3$





Prop: If $\exp_p$ is a diffeo. onto its image on $B_a(0) = \{ x \in T_pM \mid |x| < a \}$,



THEN: $\forall\, x \in B_a(0)$, the ray $x(t) = tx$, $0 \le t \le 1$

is the unique length-minimizing geodesic

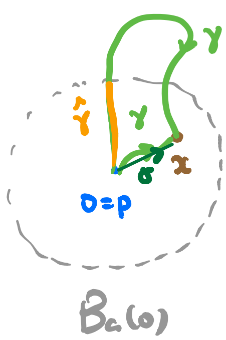
from p to $x(1)$. (ie $L(x(t)) = f(p, x(1))$)

"**Proof**": By Gauss Lemma, $\| dr \|_g \equiv 1$.

Case 1: γ stays inside $B_a(0)$

$$L(\gamma) = \int_\gamma ds \geq \int_\gamma dr = |x| = L(\sigma) < a$$

arc length



Case 2: γ leaves the ball $B_a(0)$ somewhere

$$L(\gamma) \geq L(\hat{\gamma}) \geq a$$

$\square$

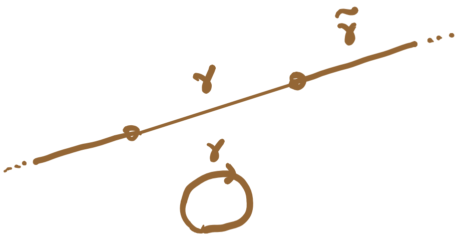
Complete Riem. manifold

Recall: $(M, g) \rightsquigarrow (M, \rho)$ metric space

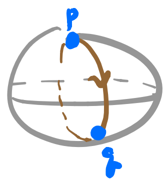
Metric space completeness: Every Cauchy seq. is convergent.

Q: How does this relate to g?

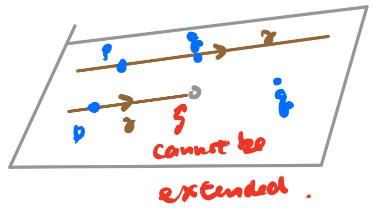
Def$^{\underline{n}}$: (M, g) geodesically complete if any geodesic in M can be extended indefinitely, i.e. $\exp_p : T_pM \to M$ is defined on the whole T_pM. $\forall p \in M$



E.g.) cpt Riem. mfd are always geod. complete



Non-E.g.: $\mathbb{R}^2 \setminus \{0\} = M$ is not complete.

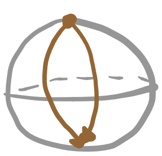


Hopf-Rinow Thm: Geodesic completeness $\Leftrightarrow$ metric completeness.

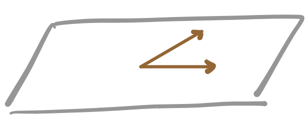
"Proof": HW exercise! [Key fact: complete $\Rightarrow \exists$ min. geod. γ joining any two points]

Q: How does the "curvatures" of (M, g) affect the behavior of geodesics?

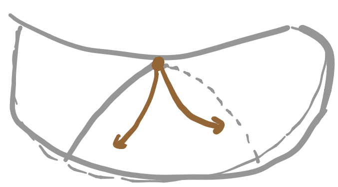
Recall: $\Sigma^2 \subseteq \mathbb{R}^3$ surface



$K > 0$



$K = 0$



$K < 0$

Idea: In higher dimensions, the sectional curvatures affects the geodesics through 2^{nd} variation of Length or "energy"

Def^n: Given a curve $\gamma : [a,b] \to (M,g)$, the **energy** of γ is:

$$E(\gamma) := \int_a^b \|\gamma'(t)\|_g^2 \, dt$$

Remarks: (1) $L(\gamma) := \int_a^b \|\gamma'(t)\|_g \, dt$ is indep of parametrization

BUT: $E(\gamma)$ depends on the parametrization as well.

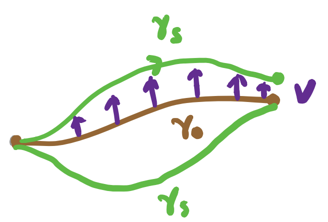
(2) $E(\gamma)$ does not involve $\sqrt{\cdot}$, so it's easier to compute its derivative.

Note: (a) Hölder $\Rightarrow$ $L(\gamma)^2 \leq (b-a)\cdot E(\gamma)$.

(b) $\rho(p,q) := \inf_\gamma L(\gamma) = \inf_\gamma E(\gamma)^{1/2}$ (Take $b-a=1$).

1st & 2nd variations of energy

Setup: Let $\gamma_s(t) : [0,1] \to M$, $s \in (-\varepsilon, \varepsilon)$, be a 1-parameter family of curves in M.



$$V(t) := \frac{\partial}{\partial s}\Big|_{s=0} \gamma_s(t)$$ variation field along γ_0

(note: $V \in \Gamma(\gamma_0^* TM)$)

$$\boxed{\frac{1}{2}\frac{d}{ds}\Big|_{s=0} E(\gamma_s) = -\int_0^1 \langle V, D_{\frac{\partial}{\partial t}} \gamma_0' \rangle_g \, dt + \langle \gamma_0', V \rangle \Big|_{t=0}^{t=1}}$$

1st var. formula for energy **Proof:** Ex!

Remark: (i) We do NOT need to assume γ_0 is p.b.a.l.

w/ fixed end pts
(ii) γ is a critical pt. of E $\Leftrightarrow$ $D_{\gamma'}\gamma' \equiv 0$, ie. γ geodesic

γ is a critical pt. of L $\Leftrightarrow$ $(D_{\gamma'}\gamma')^\perp \equiv 0$

Suppose: $\gamma = \gamma_0$ is a closed geodesic.

Then, we have the 2nd variation formula for energy:

$$\boxed{\frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} E(\gamma_s) = \int_0^1 \Big(\underbrace{\| D_{\frac{\partial}{\partial t}} V \|_g^2}_{\text{non-negative}} - \underbrace{R(\gamma', V, \gamma', V)}_{\text{sectional curvature term}} \Big)\, dt}$$

Proof: Write $F(t,s) = \gamma_s(t)$.

1st var. formula $\forall s \in (-\varepsilon,\varepsilon)$: $\frac{1}{2}\frac{d}{ds}E(\gamma_s) = -\int_0^1 \langle D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s} \rangle\, dt$ $\qquad \forall s$

$$\overset{\text{I.b.P.}}{=} \int_0^1 \langle \frac{\partial F}{\partial t}, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \rangle\, dt$$

Differentiate in s again, and take $s=0$:

$$\frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} E(\gamma_s) = \int_0^1 \Big[\langle D_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t}, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \rangle + \langle \frac{\partial F}{\partial t}, D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \rangle \Big]\bigg|_{s=0} dt$$

$\left(\begin{array}{c} \text{torsion-free} \\ \Downarrow \\ D_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial t} = D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \end{array} \right)$ $$= \int_0^1 \Big[\langle D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s}, D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \rangle + \langle \frac{\partial F}{\partial t}, D_{\frac{\partial}{\partial s}} D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial s} \rangle \Big]\bigg|_{s=0} dt$$

(using def: of R) $\longrightarrow$ $$= \int_0^1 \| D_{\frac{\partial}{\partial t}} V \|_g^2 + \langle \frac{\partial F}{\partial t}, D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s} \rangle - R\Big(\frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t}\Big)\bigg|_{s=0} dt$$

$$= \int_0^1 \| D_{\frac{\partial}{\partial t}} V \|_g^2 - R(V, \gamma', V, \gamma')\, dt$$
$$\cancel{- \int_0^1 \langle \underbrace{D_{\frac{\partial}{\partial t}} \frac{\partial F}{\partial t}}_{=0 \ \because \ \gamma_0 \text{ geodesic}}, D_{\frac{\partial}{\partial s}} \frac{\partial F}{\partial s} \rangle \bigg|_{s=0} dt} \to 0$$

$\square$

Def^{n}: $I(V,V) := \int_0^1 \left(\| D_{\frac{\partial}{\partial t}} V \|^2 - R(\gamma', V, \gamma', V) \right) dt \qquad \forall V \in \Gamma(\gamma^* TM)$

is called the index form of a geodesic γ.

Note: polarize $\rightsquigarrow$ $I(V,W)$ symmetric bilinear.

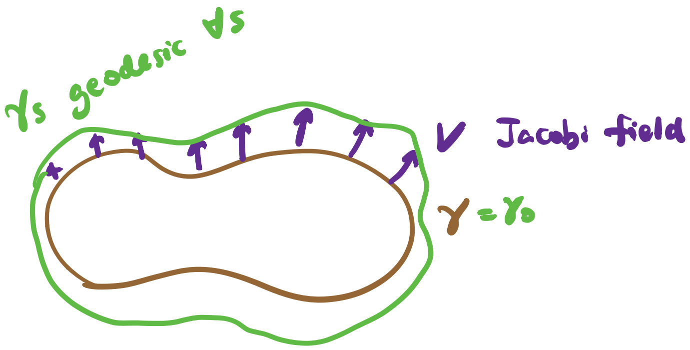
Prop: $V \in \ker I$ (ie $I(V, W) = 0 \quad \forall\, W \in T(\gamma^* TM)$)

$$\Leftrightarrow \quad \boxed{D_{\frac{\partial}{\partial t}} D_{\frac{\partial}{\partial t}} V + R(\gamma', V)\gamma' = 0}$$

Jacobi field equation (2nd order linear).

Thm: Jacobi fields V arise exactly from 1-parameter family of geodesics $\{\gamma_s\}_{s \in (-\varepsilon, \varepsilon)}$ s.t $\gamma_0 = \gamma$.

Proof: Exercise HW.



Idea: geodesics $\Leftrightarrow D_{\gamma'} \gamma' \equiv 0$
(2nd order non-linear)

Jacobi field $\Leftrightarrow$ "linearization" of $D_{\gamma'} \gamma' \equiv 0$
(2nd order linear)

$$D_{\gamma_s'} \gamma_s' \equiv 0 \qquad \forall s \in (-\varepsilon, \varepsilon).$$

$$\Rightarrow \left.\frac{d}{ds}\right|_{s=0} \left(D_{\gamma_s'} \gamma_s' \equiv 0 \right) \rightsquigarrow \text{Jacobi field eq}^{\underline{n}}$$